## Emergence of zero-lag synchronization in generic mutually coupled chaotic systems

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Zero-lag synchronization (ZLS) is achieved in a very restricted mutually coupled chaotic systems, where the delays of the self-coupling and the mutual coupling are identical or fulfil some restricted ratios. Using a set of multiple self-feedbacks we demonstrate both analytically and numerically that ZLS is achieved for a wide range of mutual delays. It indicates that ZLS can be achieved without the knowledge of the mutual distance between the communicating partners and has an important implication in the possible use of ZLS in communications networks as well as in the understanding of the emergence of such synchronization in the neuronal activities.

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Two identical chaotic systems starting from almost identical initial states, end in completely uncorrelated trajectories[1, 2]. On the other hand, chaotic systems which are mutually coupled by some of their internal variables often synchronize to a collective dynamical behavior[3, 4]. The emergence of synchronization plays important functioning roles in natural and artificial coupled systems. One of the most fascinating collective dynamical behavior is the zero-lag synchronization (ZLS), known also as an isochronal synchronization. ZLS or nearly ZLS was measured in the activity of the brain between widely separated cortical regions [5, 6, 7], where synchronization of neural activity has been shown to underlie cognitive acts[8]. The mechanism of the ZLS phenomenon has been subject of controversial debate, where the main puzzle is how two or more distant dynamical elements can synchronize at zero-lag even in the presence of non-negligible delays in the transfer of information between them.

The phenomenon of ZLS was also experimentally observed in the synchronization of two mutually chaotic semiconductor lasers, where the optical path between the lasers is a few orders of magnitude greater than the coherence length of the lasers [9, 10, 11, 12], and the analogy between the spiking optical pattern and the neuronal spiking was also recently established[13]. This phenomenon has attracted a lot of attention, mainly because of its potential for secure communication over a public channel [9]. In [14] it was recently shown that it is possible to use the ZLS phenomenon of two mutually coupled symmetric chaotic systems for a novel key-exchange protocol generated over a public-channel. Note that in contrary to a public scheme which is based on mutual coupling, private-key secure communication is based on a unidirectional coupling [15, 16] and it is susceptible to an attacker which has identical parameters and is coupled to the transmitted signal. The generation of secure communication over a public channel requires mutual coupling and was only proven to be secure based on the ZLS phenomenon[14].

Recently, it has been shown both numerically and an-

alytically that various architectures of coupled chaotic maps can exhibit ZLS[17, 18]. The main disadvantage of this phenomenon is that ZLS even between two mutually coupled chaotic systems can be achieved only for very restricted architectures and it is highly sensitive for mismatch between the delays of the mutual coupling and the self-feedback. These delays have to be identical or have to fulfil special ratios. Such a realization might exist in a time-independent point-to-point communication, but it is far from the realm of communications networks.

In this letter we first demonstrate the constraint that ZLS is achieved only for very restricted ratios between the self-feedback and the mutual delays,  $n\tau_d = m\tau_c$ , where n and m are (small) integers. We next show that one can overcome this constraint when multiple self-feedbacks are used. For the simplicity of the presentation we mainly concentrate on the Bernoulli map, where results of simulations can be compared to an analytical solution[18, 19]. However we observed the reported phenomenon for other chaotic maps and systems as well, and it is exemplified by the ZLS of mutually couple chaotic semiconductor lasers, depicted by the Lang-Kobayashi differential equations[9, 20].

The cornerstone of our system is the simplest chaotic map, the Bernoulli map, f(x) = (ax)mod1, which is chaotic for a > 1. The dynamical equations of the two mutually coupled chaotic units, X and Y, with one self-feedback (see solid lines in figure 1) are given by

$$x_{t} = (1 - \varepsilon)f(x_{t-1}) + \varepsilon[\kappa f(x_{t-\tau_{d}}) + (1 - \kappa)f(y_{t-\tau_{c}})]$$

$$y_{t} = (1 - \varepsilon)f(y_{t-1}) + \varepsilon[\kappa f(y_{t-\tau_{d}}) + (1 - \kappa)f(y_{t-\tau_{c}})]$$
(1)

where  $\tau_d$  and  $\tau_c$  are the delays of the self feedback and the mutual coupling, respectively[18]. The quantities  $1 - \varepsilon$ ,  $\varepsilon \kappa$  and  $(1 - \kappa)\varepsilon$  stand for the strength of the internal dynamics, self-feedback and the mutual coupling, respectively.

The stationary solution of the relative distance between the trajectories of the two mutually coupled chaotic Bernoulli maps can be analytically examined [18, 19]. Let us denote by  $\delta x_t$  and  $\delta y_t$  small perturbations



FIG. 1: A schematic diagram of two mutually coupled units at a distance  $\tau_c$  with one self-feedback with a delay equals to  $\tau_d$  (solid lines). Additional self-feedbacks are denoted by the dashed lines.

from the trajectories  $x_t$  and  $y_t$ , respectively. Using the ansatz  $\delta x_t = c^t \delta x_0$  and  $\delta y_t = c^t \delta y_0$  and linearizing equations (1), one can find the characteristic polynomial

$$c - a(1 - \varepsilon) - a\varepsilon\kappa c^{1 - \tau_d} + a\varepsilon(1 - \kappa)c^{1 - \tau_c} = 0$$
 (2)

where  $\lambda = \ln |c|$  is the Lyapunov exponent. Simulations of the dynamical equations (1) and the semi-analytical calculation of the maximal Lyapunov exponent of the characteristic polynomial (2), indicate that ZLS is the stationary solution of the dynamics only when the delays of the self-feedback and the mutual coupling fulfil the constraint

$$n\tau_d + m\tau_c = 0 (3)$$

where the available integers for  $n, m \in \mathbb{Z}$  are functions of  $\epsilon$  and  $\kappa$ . Results are exemplified in figure 2 for  $\epsilon = 0.9$  and  $\kappa = 0.8$  (left panel) and for  $\epsilon = 0.9$  and  $\kappa = 0.4$  (right panel). For the left panel, ZLS is achieved for the pairs (m,n) = (-1,n) where n = 1,2,...,10 and (3,-1)[21]. For the right panel ZLS is achieved for the pairs (-1,n) n = 1,...,4, (3,-1), (5,-1), (7,-1), (3,-2) and (5,-2)[22]. Those lines may have width, so the more accurate equation is  $|n\tau_d - m\tau_c| \leq \delta$ , where  $\delta \approx 2$ . The

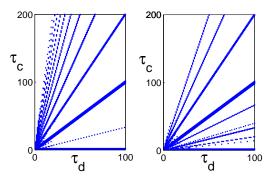


FIG. 2: Simulations and semi-analytic results for the ZLS points in the phase space  $(\tau_d, \tau_c)$  with  $a=1.1, \varepsilon=0.9$  and  $\kappa=0.8$  left panel and  $\varepsilon=0.9$  and  $\kappa=0.4$  right panel.

constraint (3) indicates that ZLS can be achieved only when  $\tau_c$  is accurately known, which is far from the realm of communications networks. In order to increase the possible ZLS range of  $\tau_c$  for a fixed  $\tau_d$ , we added more self-feedbacks, as depicted in figure 1. The generalized dynamical equations for the case of multiple self-feedbacks

are given by

$$x_{t} = (1 - \varepsilon)f(x_{t-1}) + \varepsilon \left[\kappa \sum_{l=1}^{N} \alpha_{l} f(x_{t-\tau_{d_{l}}}) + (1 - \kappa)f(y_{t-\tau_{c}})\right]$$

$$y_{t} = (1 - \varepsilon)f(y_{t-1}) + \varepsilon \left[\kappa \sum_{l=1}^{N} \alpha_{l} f(y_{t-\tau_{d_{l}}}) + (1 - \kappa)f(x_{t-\tau_{c}})\right](4)$$

where N stands for the number of self-feedbacks and the parameter  $\alpha_l$  indicates the weight of the  $l^{th}$  self-feedback fulfilling the constraint  $\sum_{l=1}^{N} \alpha_l = 1$ . In order to reveal the interplay between possible  $\tau_c$  which lead to ZLS and a given set of  $\{\tau_{d_l}\}$  we first examine in detail the case of N=2.

Results of simulations with N=2 which were confirmed by the calculation of the largest Lyapunov exponent obtained from the solution of the characteristic polynomial, similar to equation (2), are depicted in figure 3. The synchronization points  $(\tau_{d_1}, \tau_{d_2})$  where ZLS is achieved form straight lines. A careful analysis of the equations of these lines indicates that their equations are

$$n_1 \tau_{d_1} + n_2 \tau_{d_2} + m \tau_c = 0 \tag{5}$$

where  $n_1, n_2$  and m are integers. The lines may have a small width, hence a more accurate equation for the ZLS points is  $|n_1\tau_{d_1} + n_2\tau_{d_2} + m\tau_c| \leq \delta$ , where  $\delta \approx 2$ . The same equations for the ZLS lines and with similar possible width,  $\delta$ , were obtained in simulations with different  $\epsilon$ ,  $\kappa$ ,  $\alpha_i$  and  $\tau_c$ , prime and non-prime numbers in the range [31, 720].

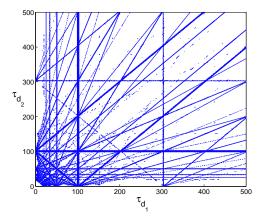
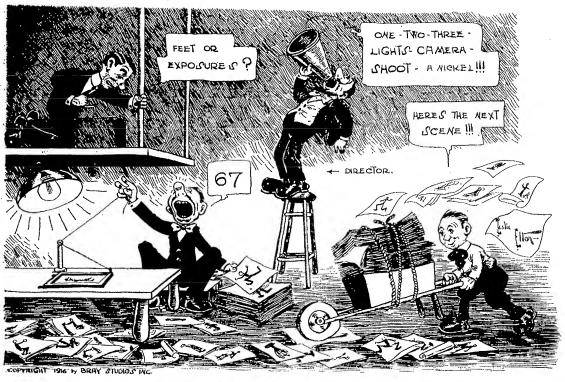


FIG. 3: Simulations and semi-analytic results for the ZLS points in the phase space  $(\tau_{d_1}, \tau_{d_2})$  for  $\tau_c = 101, \ a = 1.1, \varepsilon = 0.9, \ \kappa = 0.8$  and  $\alpha_i = 1/2$ .

Figure 3 indicates that for  $\epsilon = 0.9$  and  $\kappa = 0.8[23]$ , for instance, m can take the integers  $\pm 1$  and  $\pm 3$  only. In order to examine the possible range of the integers  $\{n_i\}$  we ran an exhaustive search simulation,  $-6 \le n_i \le 6$  and  $m = \pm 1, \pm 3$ , and obtained integer  $\tau_c$  from equation (5). Figure 4 depicts results of such an exhaustive search and



A climactic moment in the filming of a Bray thriller. Notice the drawn look on the face of the cameraman.

thousand or more of the individual pictures, but don't imagine for an instant that we draw a thousand trees. Nope. Just the same old tree put into place by what you might call a rubber-stamp method. If we want to make the tree rustle in the breeze, it takes perhaps a dozen sketches to produce the effect, but once these are done the rustling is easily accomplished by repeating these dozen over and over again in the same sequence. I am violating no confidence in telling this, for it is the only thing that makes the animated cartoon commercially possible.

Seriously, I do not believe there is any work in which as extensive results are obtained with such economy of action. Two sketches will give an effect of the briny deep scintillating under the brilliant summer sun, and a thorough-going murder can be accomplished in the most harrowing manner with less than a hundred.

Of course we seldom have murders in the cartoons, and thus far we have been able to escape the censorship in all states. The problem play has not yet reached the high point of development where it appeals to the film cartoonist, so we remain pure. There are times when we have been accused of misleading the minds of the young by

showing scenes which were unreal. I believe, however, that this is a point in our favor. Suppose the young mind is taken by its owner to consider the "Adventures of Algy" in seventy-five harrowing episodes, in each one of which the least thrilling escapade from which Algy emerges in possession of all his limbs is something like dropping from the hundredth floor of a skyscraper into a pit of boiling oil. Is there not a grave danger that the possessor of the said young mind, imbued with intense admiration for the noble Algy, may seek to emulate his example, and go diving off skyscrapers, thus mussing up the sidewalks and the pedestrians? On the other hand-consider the animated cartoon. If my hero has any such adventure as pulling the tail of a lion through the bunghole of a barrel, and tying a knot in it so that the lion cannot escape, do you think there is any danger of the young mind aforesaid being influenced to follow the hero's example? Hardly. Lions are too scarce and valuable, and their owners refuse to permit young minds to trifle with them, tease them or feed them, much less tie knots in their tails. Thus the feats the cartoon heroes perform are so unique that, while the young mind may admire their

courage and prowess, circumstances prevent them from risking their young lives in doing likewise.

Aside from these sociological aspects of the cartoons, however, it may be of interest to describe briefly the process. The first problem is that of making the movements of the figures as steady and continuous as possible. It is out of the question to draw a series of pictures of a walking man in which the movement will be as smooth as the moving picture photographs of a similar action, but this can be approximated by exercising great care. The use of tracing paper is the solution. The artist places a piece of paper upon the last drawing made, so the position last taken shows clearly, and thus he is able to make the next picture with just the sufficient variation. It is all mathematical, once the idea is planned. There is no inspiration or temperament about working in the details. We have these things figured down to milli-

So first of all I write a scenario of the cartoon and draw six to a dozen sketches of the vital points of the story—the climaxes, so to speak. Then my assistants set to work on the multiplication. But my work does not end with those original dozen drawings. Whenever there is a new action introduced, I make the sketch providing all the essentials and leave only the detail work to the staff. These drawings are then arranged in order and numbered, and all is ready for the camera.

One of the most important details then is controlling the speed of the action. This

is done by varying the number of photographs of careach toon sketch. If the scene demands that the object shall move rapidly, then slowly, then come to a stop for a moment, the pictures representing the swift action would be given one exposure each. As the tempo slows down each picture is given a correspondingly increased number of exposures. When the figure stops moving, numerous photographs are made of the same sketch, according to the time the action is suspended. As I have said, there is no guess-work about it. It is all absolutely mathematical, and we never have to make "retakes" because an actor forgot and ran when he should have walked.

The one thing about this business which is not mathematical is putting the laughs into the cartoon. This is a serious matter. There is nothing so serious as producing humor. Did you ever see a comic artist. or a writer of jokes who was not grave in demeanor, weighted down prematurely with woe and worry? The sadder a man the funnier his work, because if a thing looks funny to him it must be a scream to others. I think my present occupation as humorist to several million people weekly is due to the fact that my first experiences as an artist were so gloomy. I began life in Detroit, Michigan—no, that is not the gloomy fact to which I refer! Detroit is a lovely place, but the Detroit morgue is not especially cheerful, and the paper which employed me assigned me to the task of visiting that institution and making sketches of unidentified dead persons, victims of accidents, suicides, and such. If anyone who reads these observations has an ambition to become a humorist, I would heartily recommend him to get a job as official sketch artist in a morgue. In this

way he will pile up a sufficient store of gloom to enable him to go through life as a humorist without any necessity for renewing the supply of the stuff that makes the comics comic.

